Synthesis Imaging Theory

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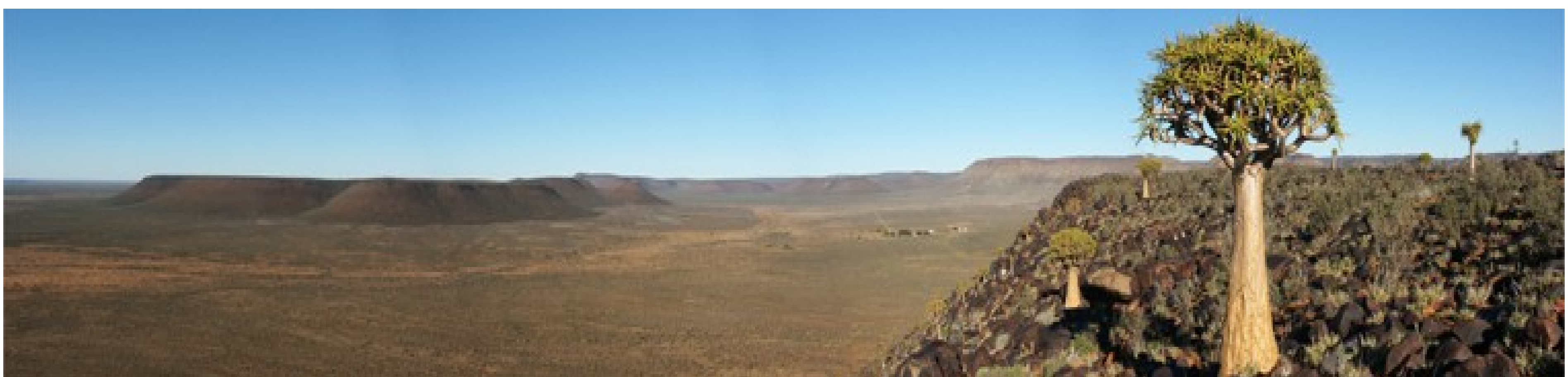


Why interferometry?

For this, diffraction theory applies – the angular resolution for a wavelength λ is :

$$\Theta \approx \lambda/D$$

In ‘practical’ units:

To obtain 1 arcsecond resolution at a wavelength of 21 cm, we require an aperture of ~42 km!

Can we synthesize an aperture of that size with pairs of antennas?

The methodology of synthesizing a continuous aperture through summations of separated pairs of antennas is called ‘aperture synthesis’.

Radio telescopes coherently sum electric fields over an aperture of size D.

We want a map

Our Goal: To measure the characteristics of celestial emission from a given direction **s**, at a given frequency ν, at a given time t.

In other words: We want a map, or image, of the emission.

Terminology/Definitions: The quantity we seek is called the brightness (or specific intensity): It is denoted here by I(**s,ν**,t), and expressed in units of: watt/(m2 Hz ster).

It is the power received, per unit solid angle from direction **s**, per unit collecting area, per unit frequency at frequency n.

Do not confuse I with Flux Density, S -- the integral of the brightness over a given solid angle:

$$S = \int I(\mathbf{s}, \nu, t)\, d\Omega$$

The units of S are: watt/(m2 Hz)

Note: 1 Jy = 10^{-26} watt/(m2 Hz).

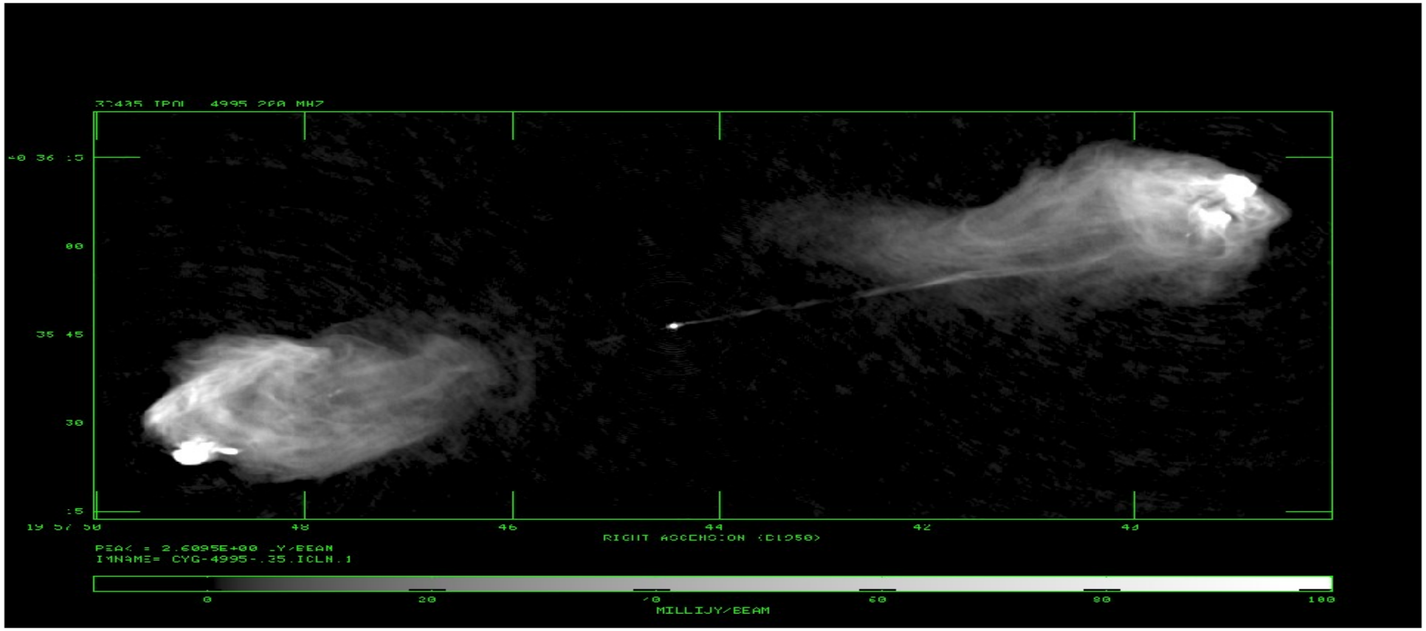
Example

I show below an image of Cygnus A at a frequency of 4995 MHz.

The units of the brightness are Jy/beam, with 1 beam = 0.16 arcsec2

The peak is 2.6 Jy/beam, which equates to 6.5 x 10-15 watt/(m2 Hz ster)

The flux density of the source is 370 Jy = 3.7 x 10^{-24} watt/(m2 Hz)



Intensity and Power.

Imagine a distant source of emission, described by brightness I(n,**s**) where **s** is a unit direction vector.

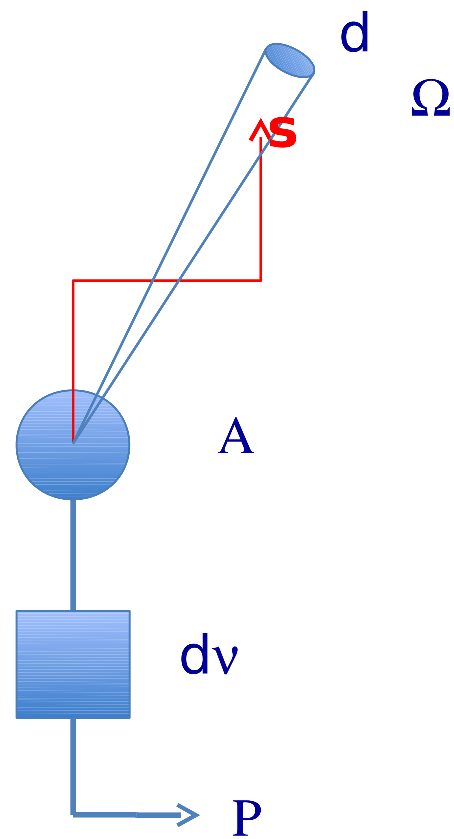
Power from this emission is intercepted by a collector (`sensor') of area A(n,**s**).

The power, P (watts) from a small solid angle dW, within a small frequency window dv, is

$$P = I(\upsilon, \mathbf{s}) A(\upsilon, \mathbf{s}) d\nu d\Omega$$

The total power received is an integral over frequency and angle, accounting for variations in the responses.

$$P = \iint I(\upsilon, \mathbf{s}) A(\upsilon, \mathbf{s}) d\upsilon d\Omega$$



Solid Angle

Sensor Area

Filter width

Power collected

The sensor

- Coherent interferometry is based on the ability to correlate

  the electric fields measured at spatially separated locations.

To do this (without mirrors) requires conversion of the electric field E(**r**,ν,t) at some place to a voltage V(ν,t) which can be conveyed to a central location for processing.

For our purpose, the sensor (a.k.a. ‘antenna’) is simply a device which senses the electric field at some place and converts this to a voltage which faithfully retains the amplitudes and phases of the electric fields.

One can imagine two kinds of ideal sensors:

An ‘all-sky’ sensor: All incoming electric fields, from all directions, are uniformly summed.

The ‘limited-field-of-view’ sensor: Only the fields from a given direction and solid angle (field of view) are collected and conveyed.

Sadly – neither of these is possible.

Quasi-monochromatic

- Analysis is simplest if the fields are perfectly monochromatic.

This is not possible – a perfectly monochromatic electric field would both have no power ($\Delta\nu = 0$), and would last forever!
So we consider instead 'quasi-monochromatic' radiation, where the bandwidth $d\nu$ is finite, but very small compared to the frequency: $d\nu << \nu$
Consider then the electric fields from a small sold angle $d\Omega$ about some direction **s**, within some small bandwidth $d\nu$, at frequency ν.
We can write the temporal dependence of this field as:

The amplitude and phase remains unchanged to a time duration of order $dt \sim 1/d\nu$, after which new values of **E** and ϕ are needed.

Simplifications

We now consider the most basic interferometer, and seek a relation between the characteristics of the product of the voltages from two separated antennas and the distribution of the brightness of the originating source emission.

To establish the basic relations, the following simplifications are introduced:

Fixed in space – no rotation or motion

Quasi-monochromatic

No frequency conversions (an ‘RF interferometer’)

Single polarization

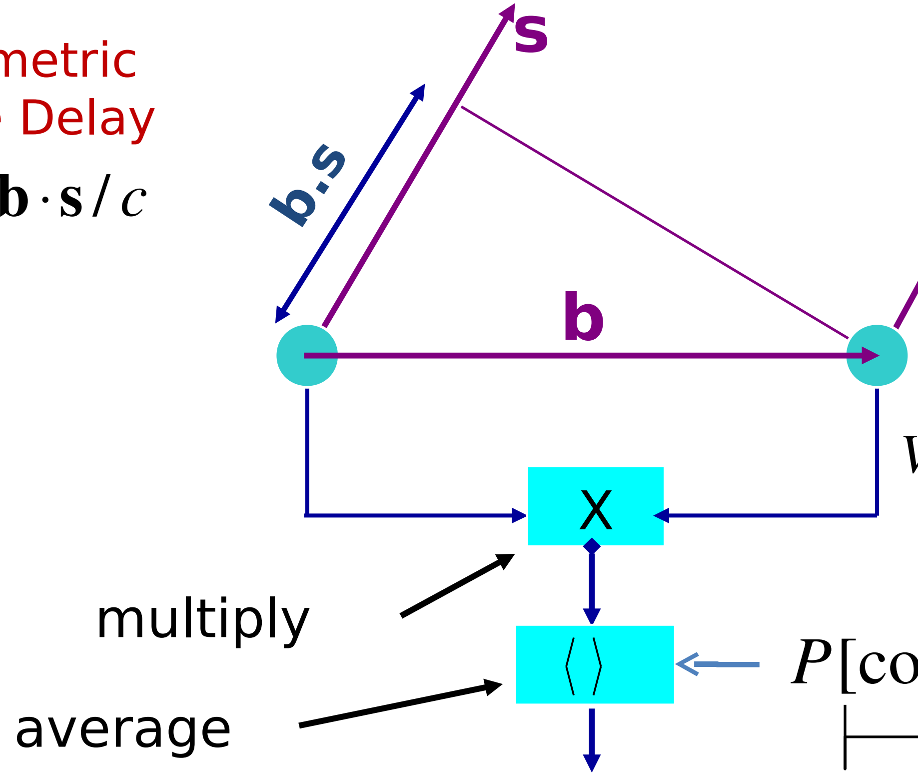
No propagation distortions (no ionosphere, atmosphere …)

Idealized electronics (perfectly linear, perfectly uniform in frequency and direction, perfectly identical for both elements, no added noise, …)

The simplest interferometer

Geometric Time Delay

$$\tau_g = \mathbf{b} \cdot \mathbf{s} / c$$



The path lengths from sensors to multiplier are assumed equal!

$$V_2 = E\cos(\omega t)$$

$$P[\cos(\omega\tau_g) + \cos(2\omega t - \omega\tau_g)]$$

Unchanging

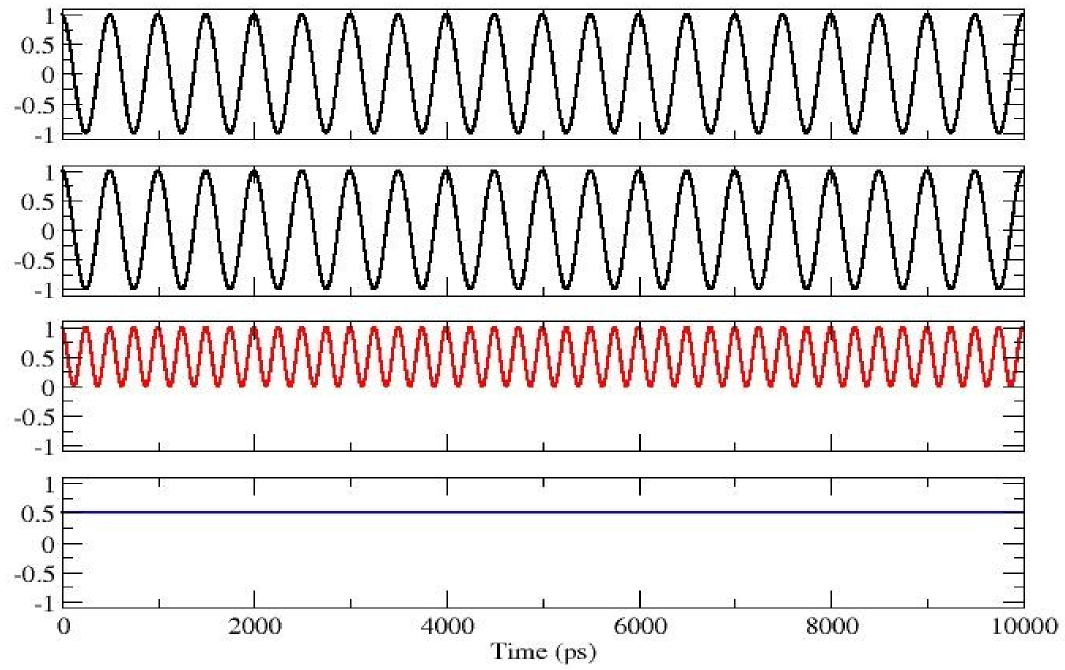
Rapidly varying,

Example -in phase

2 GHz Frequency, with voltages in phase:

$\mathbf{b}.\mathbf{s} = n\lambda$, or $\tau_g = n/\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average

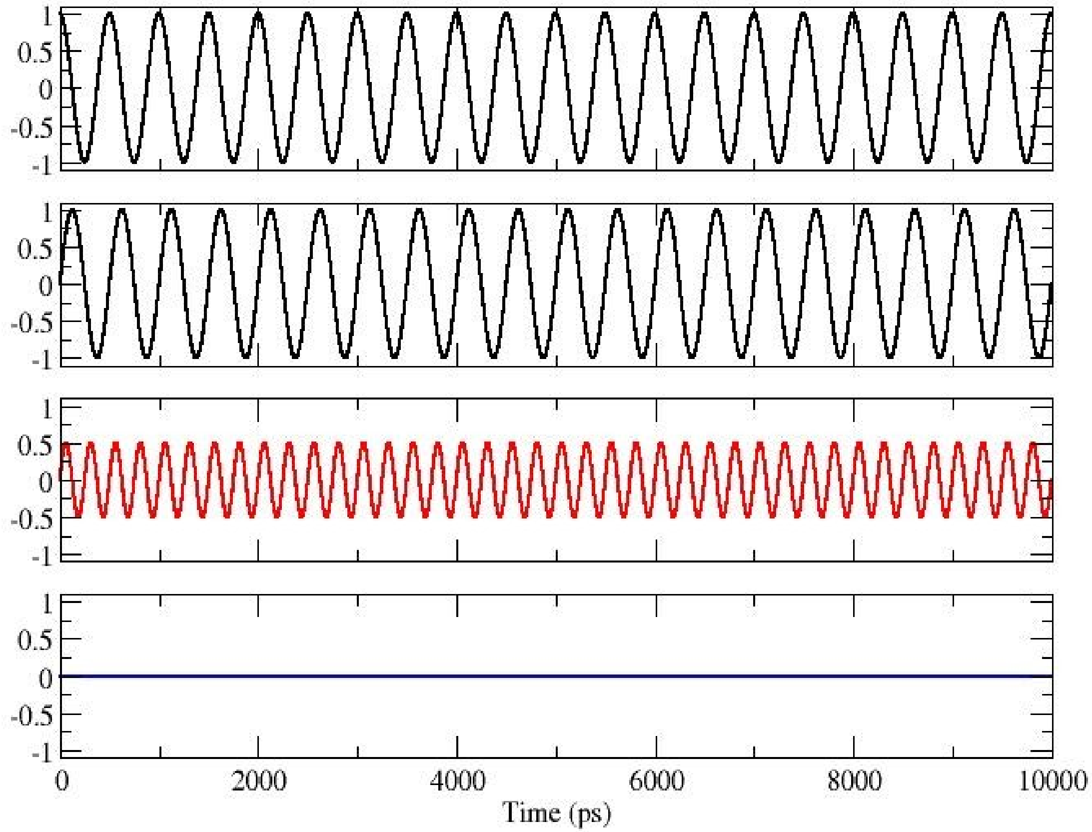


Example -in quadrature

2 GHz Frequency, with voltages in quadrature phase:

$b.s=(n +/- \frac{1}{4})\lambda$, $\tau g = (4n +/- 1)/4\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average

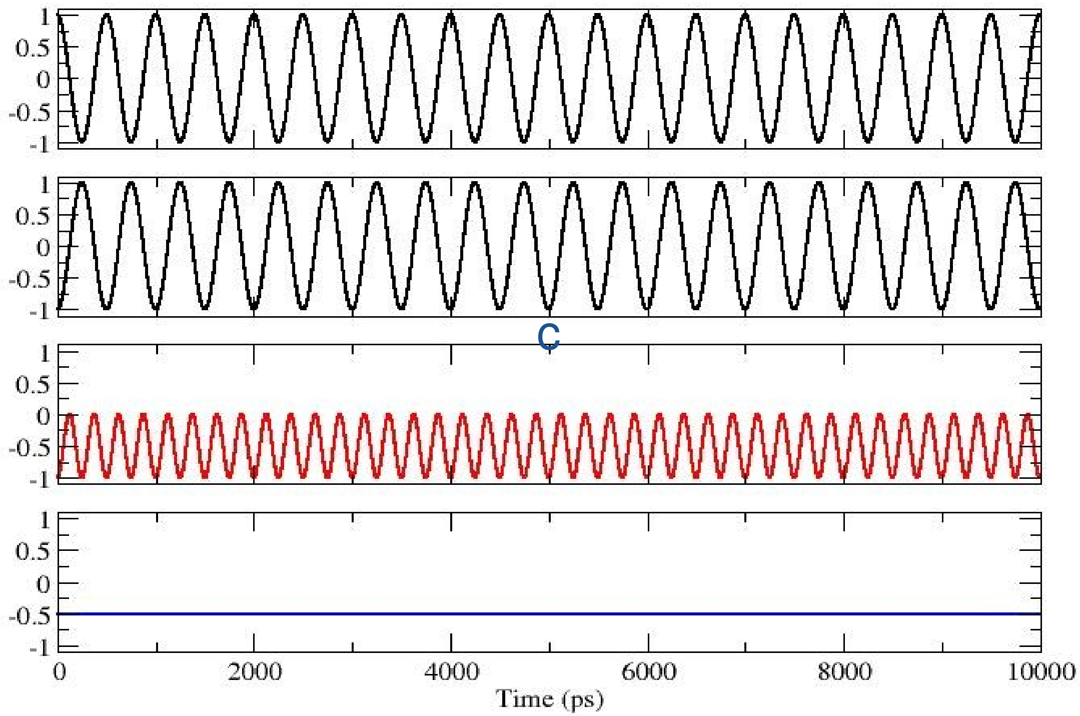


Example -in antiphase

2 GHz Frequency, with voltages out of phase:

$b.s=(n \pm \frac{1}{2})\lambda$ $\tau g = (2n \pm 1)/2\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average



The averaged product RC is dependent on the received power, $P = E^2/2$ and geometric delay, tg, and hence on the baseline orientation and source direction:

$$\omega\,\tau_g = 2\pi\nu\,\mathbf{b.s}/c = 2\pi\,\mathbf{b.s}/\lambda$$

Note that RC is not a a function of:

The time of the observation -- provided the source itself is not variable!

The location of the baseline -- provided the emission is in the far-field.

The actual phase of the incoming signal – the distance of the source does not matter, provided it is in the far-field.

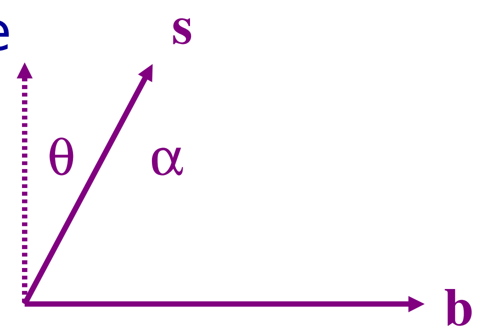
1D Example

To illustrate the response, expand the dot product in one dimension:

Here, **$u = b/\lambda$ is the baseline length in wavelengths**, and θ is the angle w.r.t. the plane perpendicular to the baseline.

$l = \cos(\alpha) = \sin(\theta)$

is the direction cosine



Consider the response Rc, as a function of angle, for two different baselines with u = 10, and u = 25 wavelengths:

$$R_C = \cos(2\pi u l)$$

Whole-Sky Response

Top:

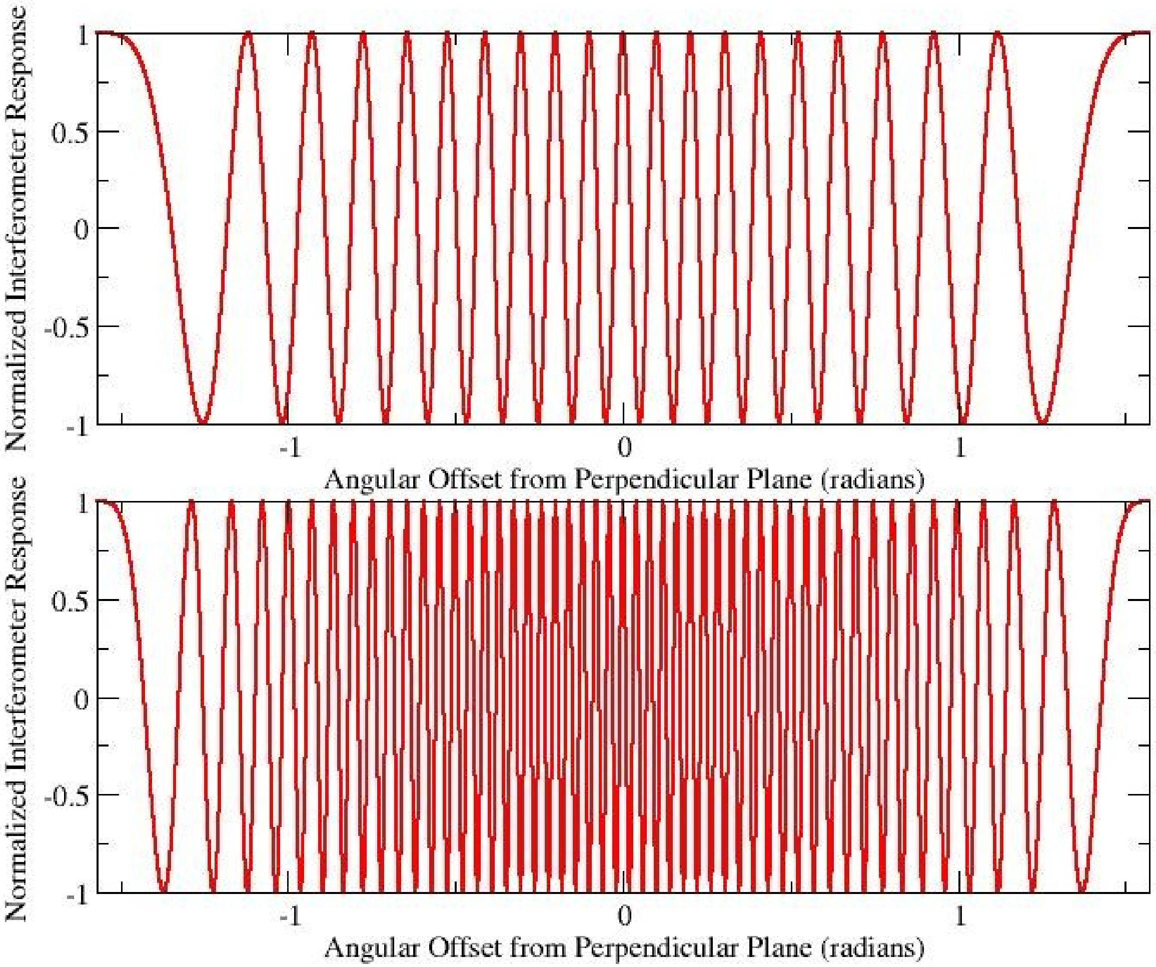
u = 10

There are 20 whole fringes over the hemisphere.

Bottom:

u = 25

There are 50 whole fringes over the hemisphere

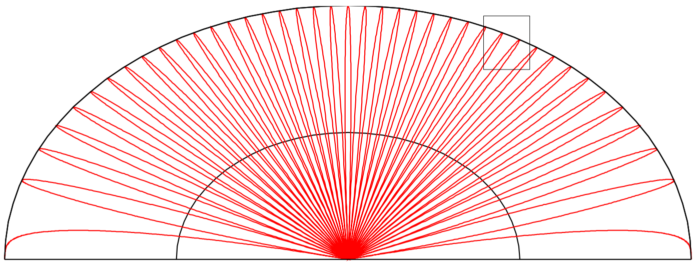


From an Angular Perspective

Top Panel:

The absolute value of the response for u = 10, as a function of angle.

The ‘lobes’ of the response pattern alternate in sign.

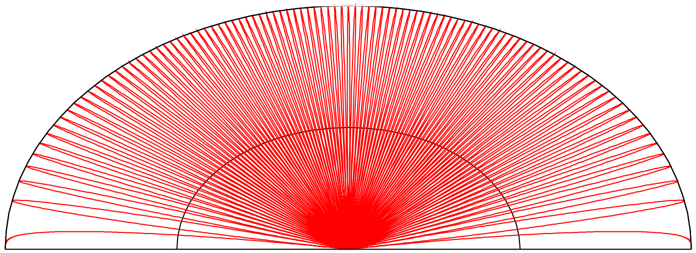


Bottom Panel:

The same, but for u = 25.

Angular separation between lobes (of the same sign) is

$\delta\theta \sim 1/u = \lambda/b$ radians.



Hemispheric Pattern

The preceding plot is a meridional cut through the hemisphere, oriented along the baseline vector.

In the two-dimensional space, the fringe pattern consists of a series of coaxial cones, oriented along the baseline vector.

The figure is a two-dimensional representation when u = 4.

As viewed along the baseline vector, the fringes show a 'bulls-eye' pattern – concentric circles.

The Effect of the Sensor

The patterns shown presume the sensor has isotropic response.

This is a convenient assumption, but (sadly, in some cases) doesn’t represent reality.

Real sensors impose their own patterns, which modulate the amplitude and phase, of the output.

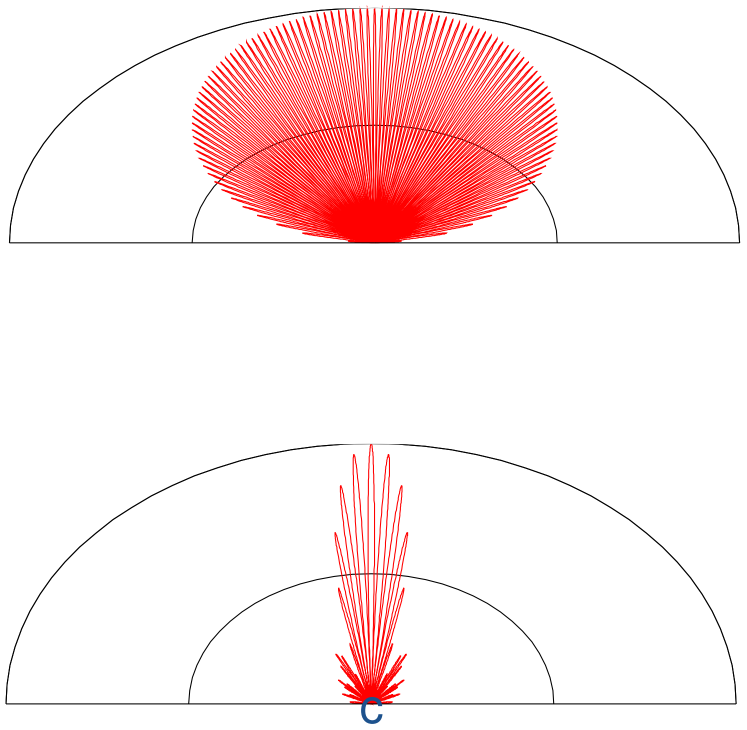
Large sensors (a.k.a. ‘antennas’) have very high directivity --very useful for some applications.

The Effect of Sensor Patterns

Sensors (or antennas) are not isotropic, and have their own responses.

Top Panel: The interferometer pattern with a cos(θ)-like sensor response.

Bottom Panel: A multiple-wavelength aperture antenna has a narrow beam, but also sidelobes.



Extended Source response

- The response from an extended source is obtained by summing the responses for each antenna over the sky, multiplying, and averaging:

$$R_c = \langle \int V_1 d\Omega_1 \int V_2 d\Omega_2 \rangle$$

- The expectation, and integrals can be interchanged, and providing the emission is spatially incoherent, we get

$$R_c = \int\int I_\nu (s) \cos(2\pi\nu \, \mathbf{b} \cdot \mathbf{s} / c) \, d\Omega$$

- This expression links what we want – the source brightness on the sky, $I_\nu(s)$, – to something we can measure - RC, the interferometer response.

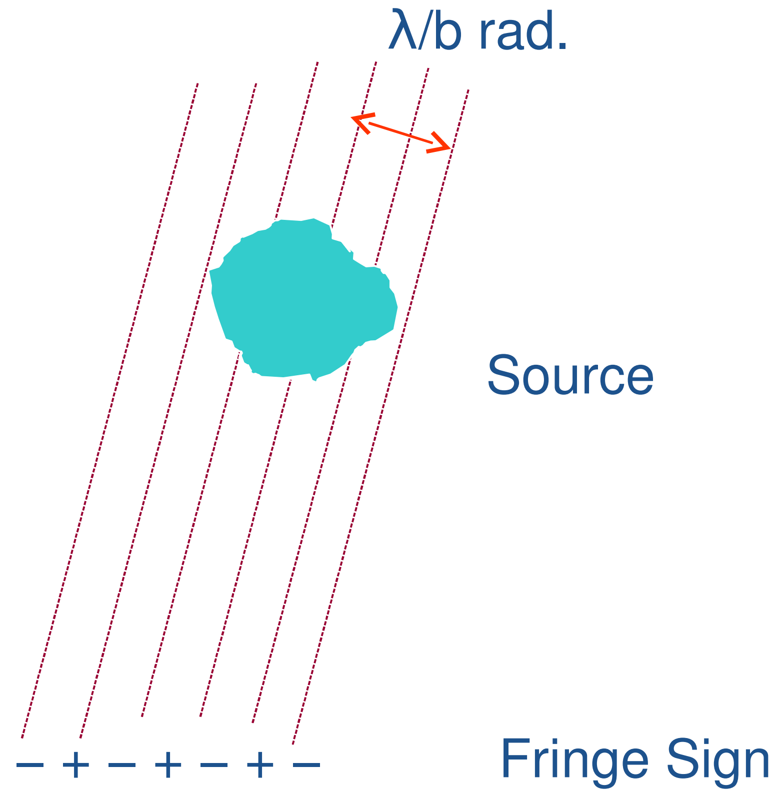
Schematic Illustration

- The correlator can be thought of 'casting' a sinusoidal coherence pattern, of angular scale λ/b radians, onto the sky.
- The correlator multiplies the source brightness by this coherence pattern, and integrates (sums) the result over the sky.

- Orientation set by baseline geometry.
- Fringe separation set by

  (projected) baseline length and wavelength.

  - Long baseline gives close-packed fringes
  - Short baseline gives widely-separated fringes
- Physical location of baseline unimportant, provided source is in the far field.



Odd and Even functions

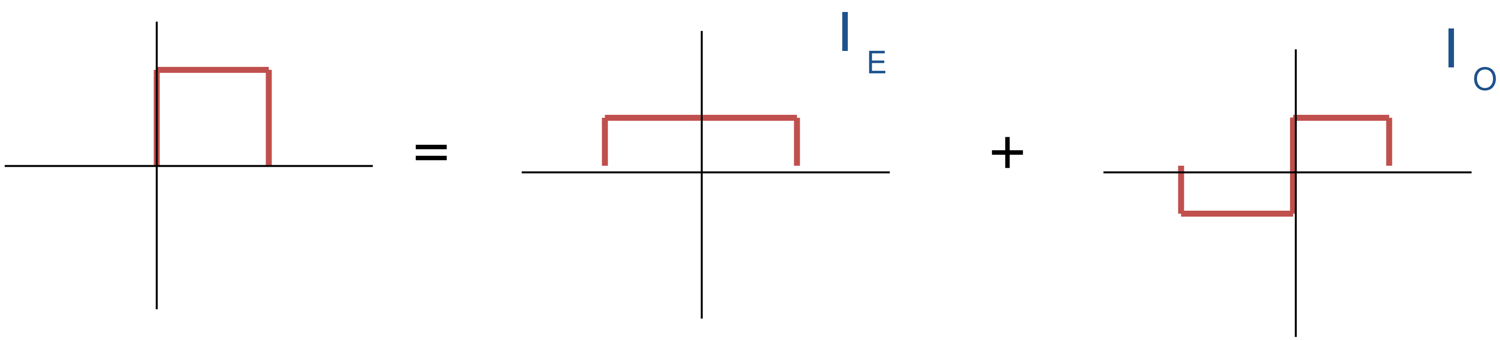
- But the measured quantity, Rc, is insufficient – it is only sensitive to the ‘even’ part of the brightness, IE(s).
- Any real function, I(x,y), can be expressed as the sum of two real functions which have specific symmetries:

An even part:

$$I_E(x, y) = ½ (I(x, y) + I(-x, -y)) = I_E(-x, -y)$$

An odd part:

$$I_O(x, y) = ½ (I(x, y) - I(-x, -y)) = -I_O(-x, -y)$$



But One Correlator is Not Enough!

The correlator response, Rc:

$$R_C = \iint I_\nu(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega$$

is not enough to recover the correct brightness. Why?

Suppose that the source of emission has a component with odd symmetry:

$$I_o(s) = -I_o(-s)$$

Since the cosine fringe pattern is even, the response of our interferometer to the odd brightness distribution is 0!

$$R_c = \iint I_o(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega \;= 0$$

Hence, we need more information if we are to completely recover the source brightness.

Why Two Correlations are Needed

The integration of the cosine response, Rc, over the source brightness is sensitive to only the even part of the brightness:

$$R_C = \iint I(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega = \iint I_E(\mathbf{s})\cos(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega$$

since the integral of an odd function (IO) with an even function (cos x) is zero.

To recover the 'odd' part of the intensity, IO, we need an 'odd' fringe pattern. Let us replace the 'cos' with 'sin' in the integral

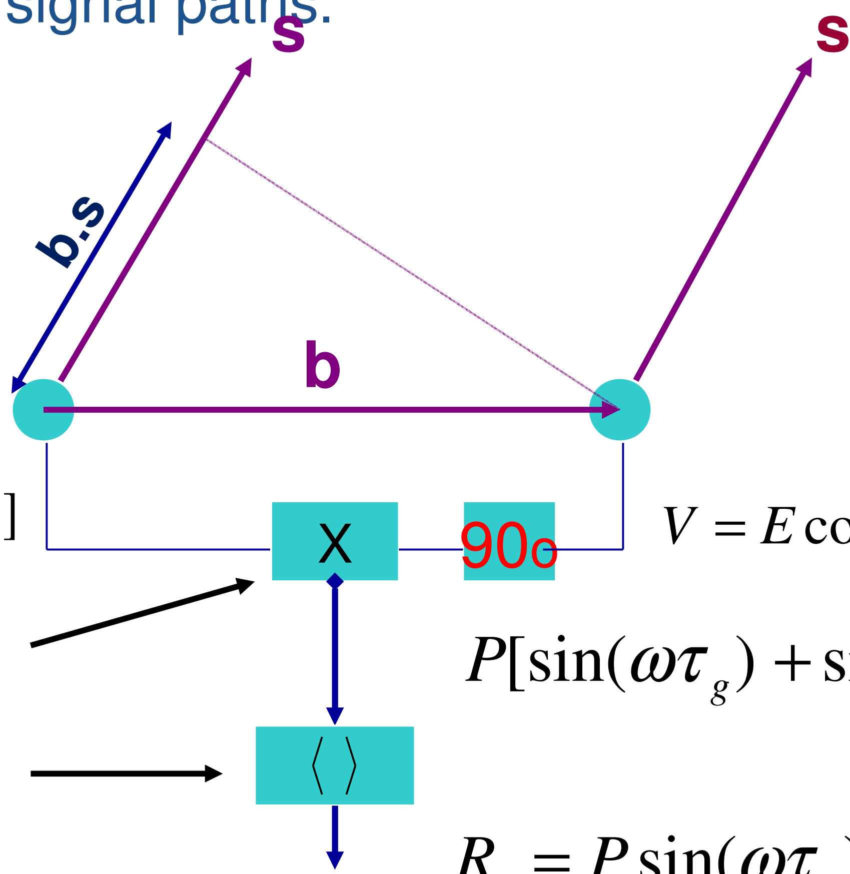
$$R_S = \iint I(\mathbf{s})\sin(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega = \iint I_O(\mathbf{s})\sin(2\pi\nu\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega$$

since the integral of an even times an odd function is zero.

Making a SIN Correlator

We generate the 'sine' pattern by inserting a 90 degree phase shift in one of the signal paths.

$$\tau_g = \mathbf{b} \cdot \mathbf{s} / c$$



$$V = E\cos[\omega(t - \tau_g)]$$

$$V = E\cos(\omega t)$$

multiply

average

$$P[\sin(\omega\tau_g) + \sin(2\omega t - \omega\tau_g)]$$

$$R_s = P\sin(\omega\tau_g)$$

Define the Complex Visibility

We now DEFINE a complex function, the complex visibility, V, from the two independent (real) correlator outputs RC and RS:

$$V = R_C - iR_S = Ae^{-i\phi}$$

where

$$A = \sqrt{R_C^2 + R_S^2}$$

$$\phi = \tan^{-1}\left(\frac{R_S}{R_C}\right)$$

This gives us a beautiful and useful relationship between the source brightness, and the response of an interferometer:

$$V_\nu(\mathbf{b}) = R_C - iR_S = \iint I_\nu(\mathbf{s}) e^{-2\pi i\nu\mathbf{b}\cdot\mathbf{s}/c}\, d\Omega$$

Under some circumstances, this is a 2-D Fourier transform, giving us a well established way to recover I(s) from V(b).

We now DEFINE a complex function, the complex visibility, V, fro
independent (real) correlator outputs RC and RS:

The Complex Correlator and Complex Notation

A correlator which produces both 'Real' and 'Imaginary' parts – or the Cosine and Sine fringes, is called a 'Complex Correlator'

For a complex correlator, think of two independent sets of projected sinusoids, 90 degrees apart on the sky.

In our scenario, both components are necessary, because we have assumed there is no motion – the 'fringes' are fixed on the source emission, which is itself stationary.

The complex output of the complex correlator also means we can use complex analysis throughout: Let:

$$V_1 = A\cos(\omega t) = \mathrm{Re}\left(A e^{-i\omega t}\right)$$

$$V_2 = A\cos[\omega(t - \mathbf{b}.\mathbf{s}/c)] = \mathrm{Re}\left(A e^{-i\omega(t-b\bullet s/c)}\right)$$

Then:

$$P_{corr} = \left\langle V_1 V_2^* \right\rangle = P e^{-i\omega\, \mathbf{b}\bullet\mathbf{s}/c}$$

Picturing the Visibility

The source brightness is Gaussian, shown in black.

The interferometer ‘fringes’ are in red.

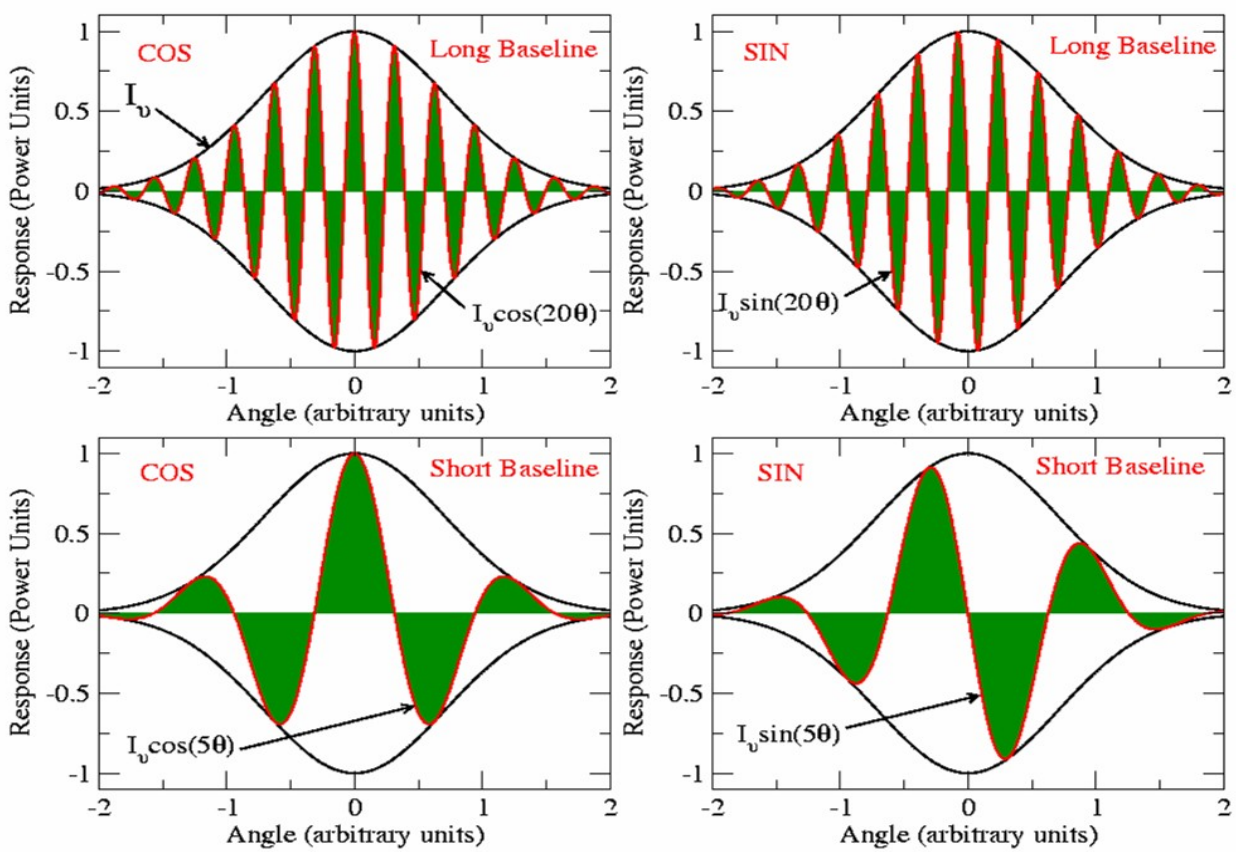
The visibility is the integral of the product – the net dark green area.

RC

RS

Long Baseline

Short Baseline



Examples of 1-D Visibilities

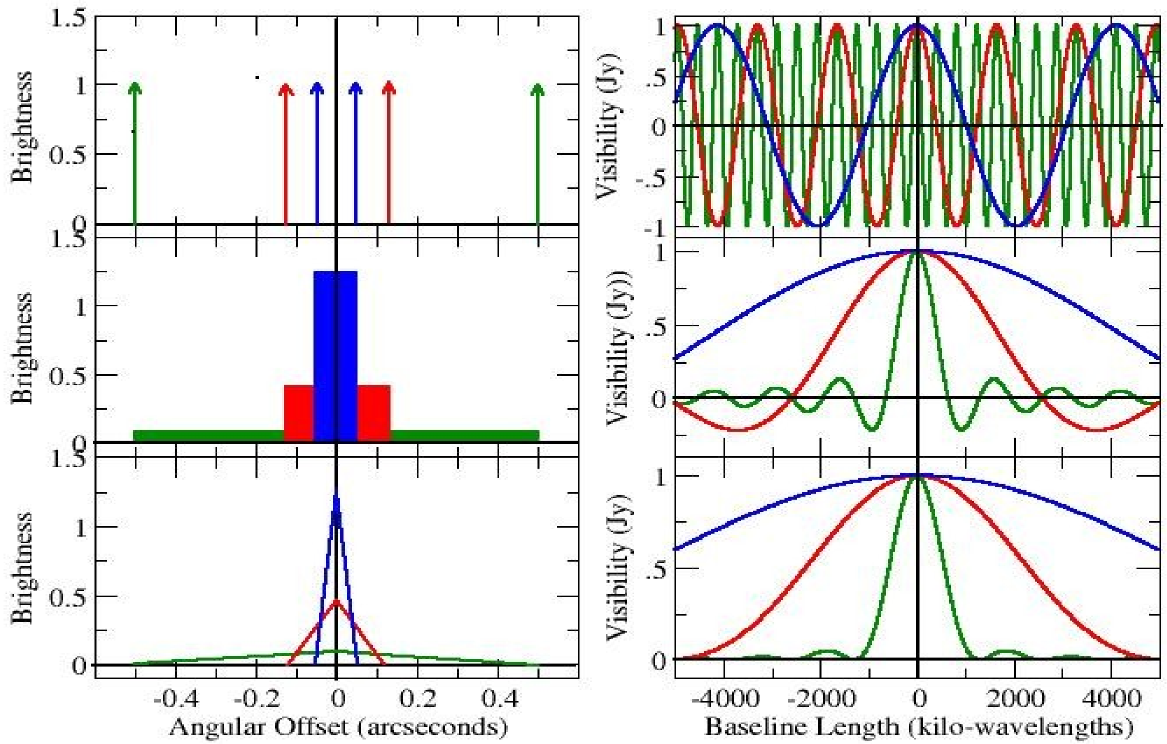
Simple pictures are easy to make illustrating 1-dimensional visibilities.

Brightness Distribution Visibility Function

Unresolved Doubles

Uniform

Central Peaked



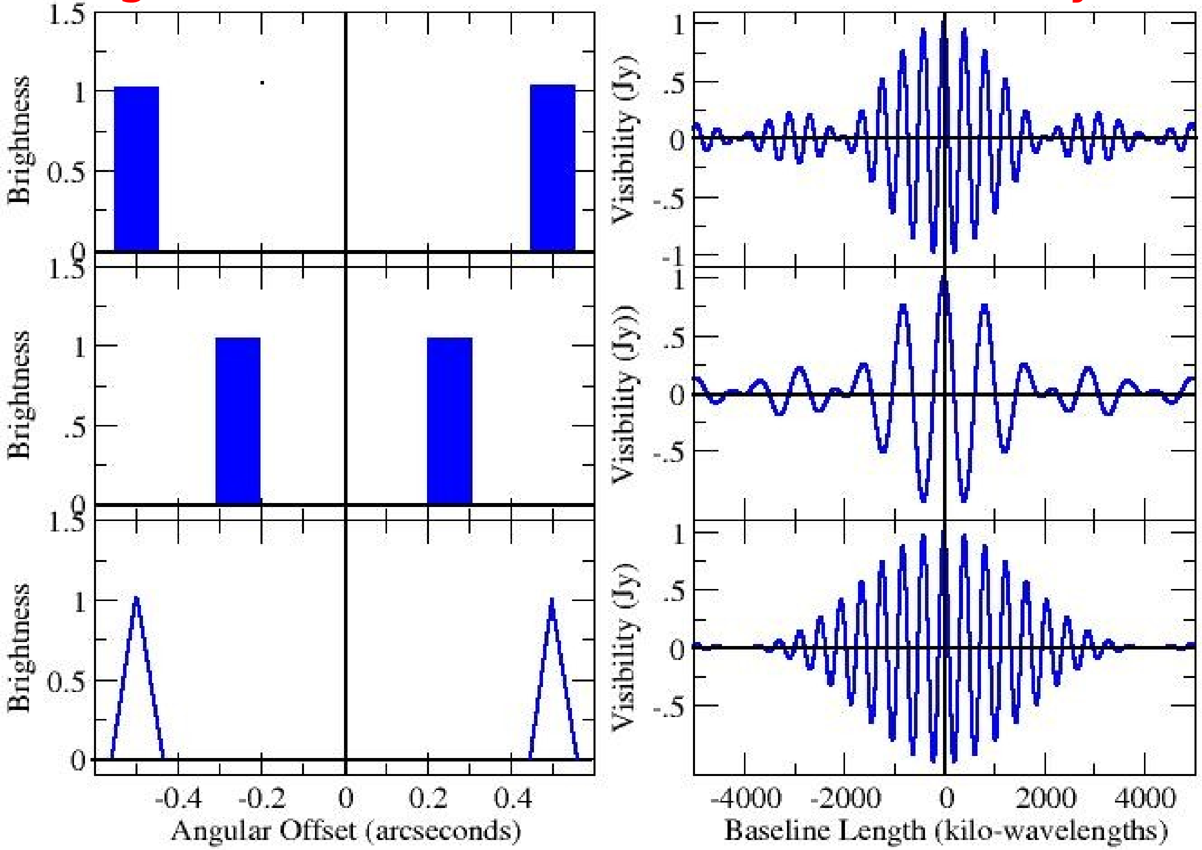
More Examples

- Simple pictures are easy to make illustrating 1-dimensional visibilities.

Brightness Distribution Visibility

Function

- Resolved Double
- Resolved Double
- Central Peaked Double



Basic Characteristics of the Visibility

For a zero-spacing interferometer, we get the 'single-dish' (total-power) response.
As the baseline gets longer, the visibility amplitude will in general decline.
When the visibility is close to zero, the source is said to be 'resolved out'.
Interchanging antennas in a baseline causes the phase to be negated – the visibility of the 'reversed baseline' is the complex conjugate of the original.
Mathematically, the visibility is Hermitian, because the brightness is a real function.

The Visibility is a unique function of the source brightness.
The two functions are related through a Fourier transform.

$$V(u,v) \Leftrightarrow I(l,m)$$

An interferometer, at any one time, makes one measure of the visibility, at baseline coordinate (u,v).

Sufficient knowledge of the visibility function (as derived from an interferometer) will provide us a reasonable estimate of the source brightness.

How many is ‘sufficient’, and how good is ‘reasonable’?

These simple questions do not have easy answers…

Comments on the Visibility

The Visibility is a function of the source structure and the interferometer baseline length and orientation.

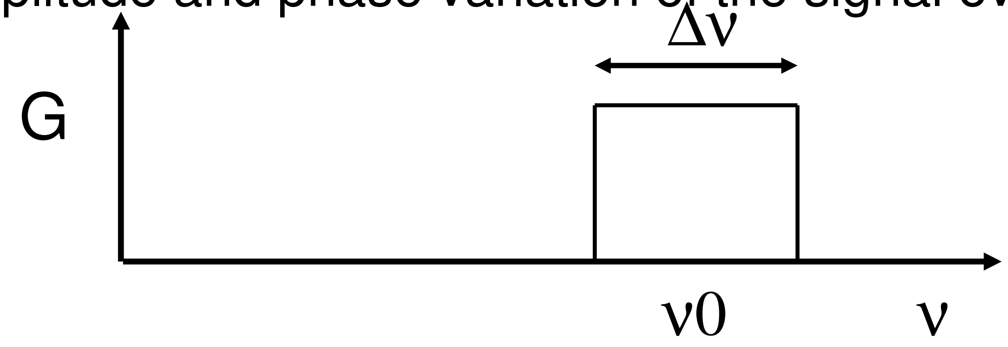
Each observation of the source with a given baseline length and orientation provides one measure of the visibility.

Sufficient knowledge of the visibility function (as derived from an interferometer) will provide us a reasonable estimate of the source brightness.

The Effect of Bandwidth.

Real interferometers must accept a range of frequencies. So we now consider the response of our interferometer over frequency.

To do this, we first define the frequency response functions, G(v), as the amplitude and phase variation of the signal over frequency.



The function G(v) is primarily due to the gain and phase characteristics of the electronics, but can also contain propagation path effects.

The Effect of Bandwidth.

To find the finite-bandwidth response, we integrate our fundamental response over a frequency width $\Delta\nu$, centered at ν_0:

$$V = \int \left(\frac{1}{\Delta\nu} \int_{\upsilon_0-\Delta\upsilon/2}^{\upsilon_0+\Delta\upsilon/2} I(\mathbf{s},\upsilon) G_1(\upsilon) G_2^*(\upsilon) e^{-i2\pi\upsilon\tau_g} \, d\upsilon \right) d\Omega$$

If the source intensity does not vary over the bandwidth, and the instrumental gain parameters G are square and real, then

$$V = \int \left(\frac{1}{\Delta\nu} \int_{\upsilon_0-\Delta\upsilon/2}^{\upsilon_0+\Delta\upsilon/2} I(\mathbf{s},\upsilon) G_1(\upsilon) G_2^*(\upsilon) e^{-i2\pi\upsilon\tau_g} \, d\upsilon \right) d\Omega$$

where the **fringe attenuation function**, sinc(x), is defined as:

The Bandwidth/FOV limit

This shows that the source emission is attenuated by the spatially variant function sinc(x), also known as the 'fringe-washing' function.

The attenuation is small when:

$$\tau_g \Delta \upsilon << 1$$

which occurs when the source offset θ ιs less than: (exercise for the student)

$$\theta << \frac{\lambda}{b} \frac{\upsilon_0}{\Delta \upsilon} = \theta_{res} \frac{\upsilon_0}{\Delta \upsilon}$$

The ratio $\nu 0/\Delta\nu$ is the inverse fractional bandwidth – for the EVLA, this ratio is never less than ~500.

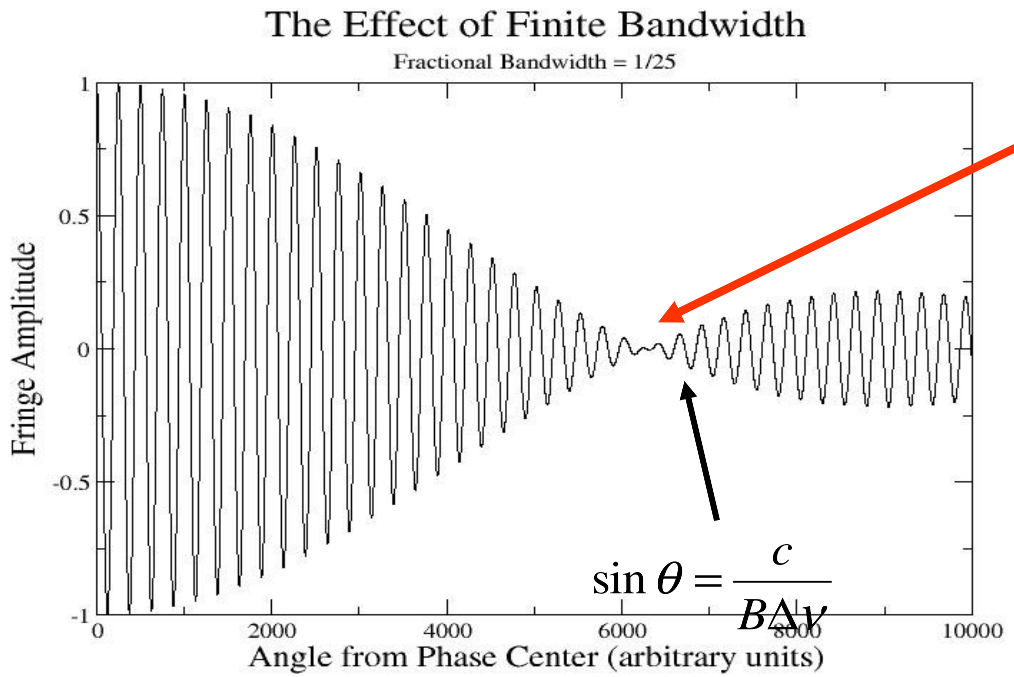
The fringe attenuation is infinite (i.e. no response) when

$$\sin \theta = \frac{c}{B \Delta \upsilon}$$

Bandwidth Effect Example

For a square bandpass, the bandwidth attenuation reaches a null at an angle equal to the fringe separation divided by the fractional bandwidth: $\Delta\nu/\nu_0$

If $\Delta\nu$ = 2 MHz, and B = 35 km, then the null occurs at about 27 degrees off the meridian. (Worst case for EVLA).



$$\sin\theta = \frac{c}{B\Delta\nu}$$

Fringe Attenuation function:

$$\mathrm{sinc}\left(\frac{B}{\lambda}\frac{\Delta\upsilon}{\upsilon}\theta\right)$$

Note: The fringe-washing function depends only on bandwidth and baseline – not on frequency.

Observations off the Meridian

In our basic scenario (stationary source, stationary interferometer), the effect of finite bandwidth can strongly attenuate the visibility from sources far from the meridional plane. Suppose we wish to observe an object far from that plane?

One solution is to use a very narrow bandwidth – this loses sensitivity, which can only be made up by utilizing many channels – feasible, but computationally expensive.

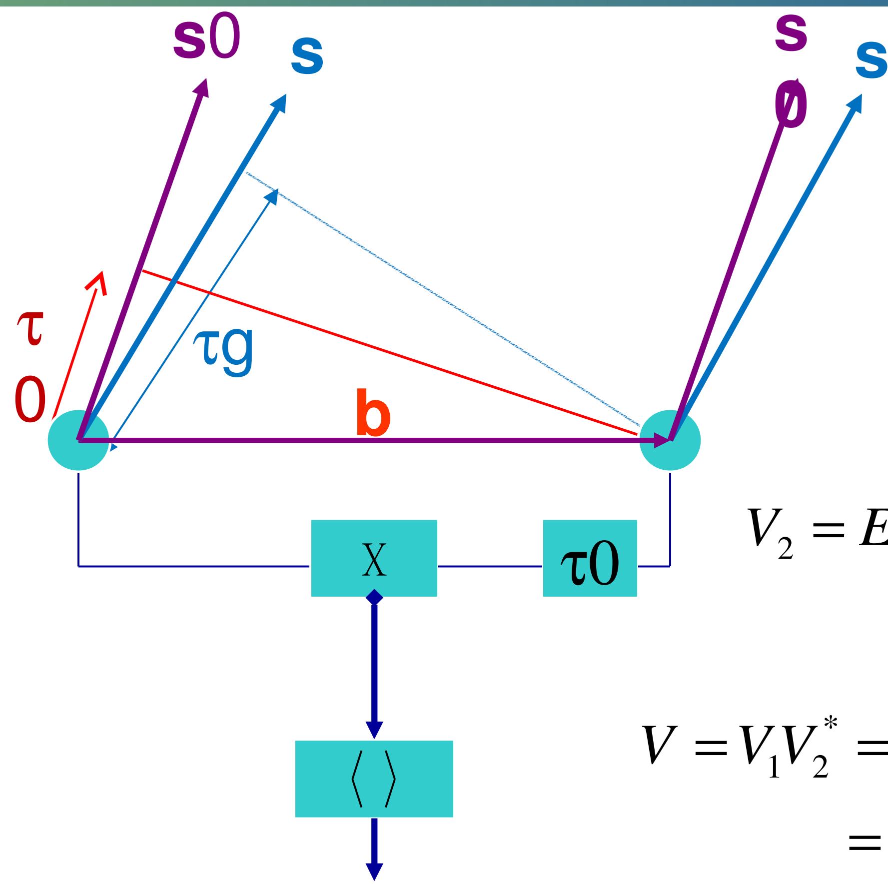
Better answer: Shift the fringe-attenuation function to the center of the source of interest.

-Delay compensation

Adding Time Delay

$$\mathrm{sinc}\left(\frac{B}{\lambda}\frac{\Delta\upsilon}{\upsilon}\theta\right)$$

$$\tau_0 = \mathbf{b}\cdot\mathbf{s}_0 / c$$



S0 = reference direction

S = general direction

$$V_1 = E e^{i\omega(t-\tau_g)}$$

$$V_2 = E e^{i\omega(t-\tau_0)}$$

$$V = V_1 V_2^* = E^2 e^{-i[\omega(\tau_g - \tau_0)]}$$

$$= E^2 e^{-i2\pi[\upsilon \mathbf{b}\cdot(\mathbf{s}-\mathbf{s}_0)/c]}$$

The entire fringe pattern has been shifted over by angle

sin θ = cτ0/b

Observations from a Rotating Platform

Real interferometers are built on the surface of the earth – a rotating platform. From the observer's perspective, sources move across the sky.

Since we know how to adjust the interferometer to move its coherence pattern to the direction of interest, it is a simple step to continuously move the pattern to follow a moving source.

All that is necessary is to continuously slip the inserted time delay, with an accuracy $\delta\tau << 1/\Delta\nu$ to minimize bandwidth loss.

For the 'radio-frequency' interferometer we are discussing here, this will automatically track both the fringe pattern and the fringe-washing function with the source.

Hence, a point source, at the reference position, will give uniform amplitude and zero phase throughout time (provided real-life things like the atmosphere, ionosphere, or geometry errors don't mess things up …)

Time Averaging Loss

So – we can track a moving source, continuously adjusting the delay, to prevent bandwidth losses.

This also ‘moves’ the cosinusoidal fringe pattern – very convenient!

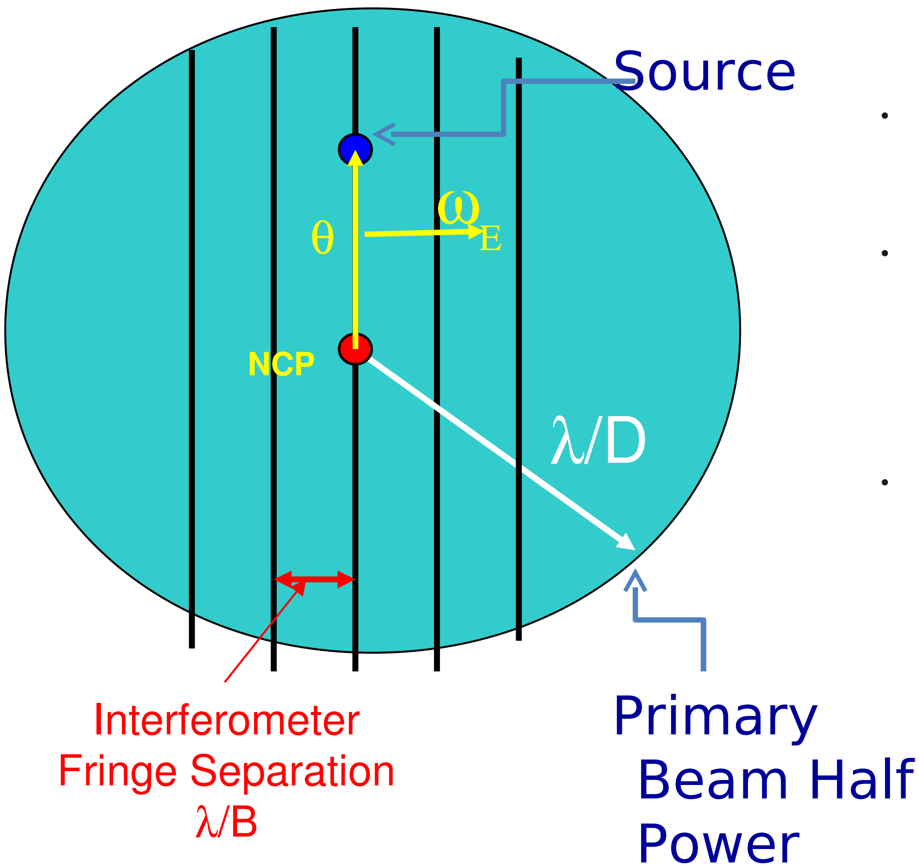
From this, you might think that you can increase the time averaging for as long as you please.

But you can’t – because the convenient tracking only works perfectly for the object ‘in the center’.

All other sources are moving w.r.t. the fringes …

Time-Smearing Loss Timescale

Simple derivation of fringe period, from observation at the NCP.



- Turquoise area is antenna primary beam on the sky – radius = λ/D
- Interferometer coherence pattern has spacing = λ/B
- Sources in sky rotate about NCP at angular rate:

  ω_{ε} =7.3x10-5 rad/sec.

- Minimum time taken for a source to move by λ/B at angular distance θ is:

$$t = \frac{\lambda}{B}\frac{1}{\omega_E \theta} \approx \frac{D}{\omega_E B}$$

Time-Averaging Loss

In our scenario moving sources and a 'radio frequency' interferometer, adding time delay to eliminate bandwidth losses also moves the fringe pattern.

A major advantage of 'tracking' the target source is that the rate of change of visibility phase is greatly decreased – allowing us to integrate longer, and hence reduce database size.

How long can you integrate before the differential motion shifts the source through the fringe pattern?

Worst case: (whole hemisphere): $t = \lambda/(B\omega_E)$ sec = 83 msec at 21 cm.

Worst case for EVLA: $t = D/(B\omega_E)$ = 10 seconds. (A-config., max. baseline)

To prevent 'delay losses', your averaging time must be much less than this.

The Heterodyne Interferometer: LOs, IFs, and Downcoversion

This would be the end of the story (so far as the fundamentals are concerned) if all the internal electronics of an interferometer would work at the observing frequency (often called the 'radio frequency', or RF).

Unfortunately, this cannot be done in general, as high frequency components are much more expensive, and generally perform more poorly than low frequency components.

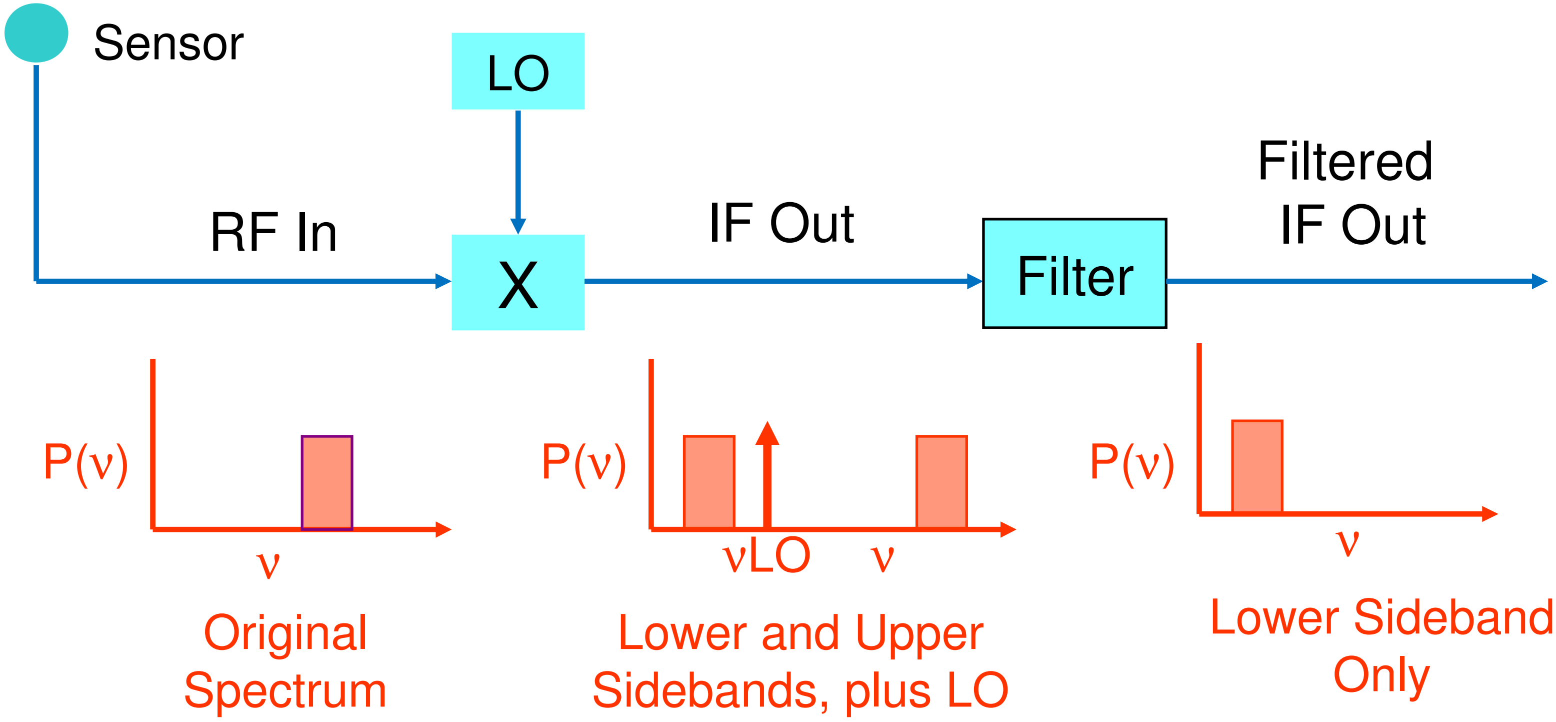
Thus, most radio interferometers use 'down-conversion' to translate the radio frequency information from the 'RF', to a lower frequency band, called the 'IF' in the jargon of our trade.

For signals in the radio-frequency part of the spectrum, this can be done with almost no loss of information.

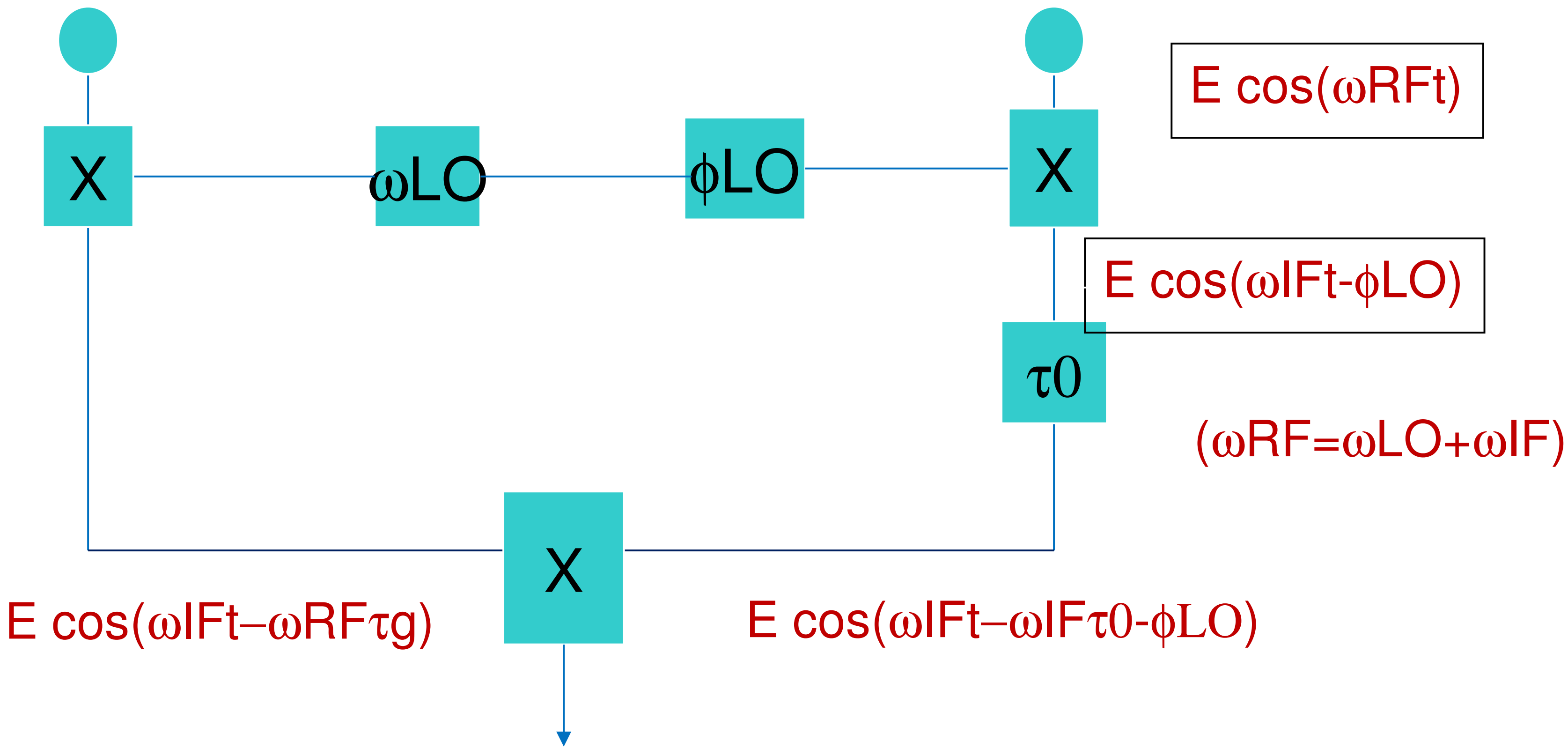
But there is an important side-effect from this operation in interferometry, which we now review.

Downconversion

At radio frequencies, the spectral content within a passband can be shifted – with almost no loss in information, to a lower frequency through multiplication by a 'LO' signal.



Signal Relations, with LO Downconversion



$$V = E^2 e^{-i(\omega_{RF}\tau_g - \omega_{IF}\tau_0 - \phi_{LO})}$$

Recovering the Correct Visibility Phase

The correct phase is: $\omega_{RF}(\tau_g - \tau_0)$.

The observed phase is: $\omega_{RF}\tau_g - \omega_{IF}\tau_0 - \phi_{LO}$

These will be the same when the LO phase is set to:

$$\phi_{LO} = \omega_{LO}\tau_0$$

This is necessary because the delay, τ_0, has been added in the IF portion of the signal path, rather than at the frequency at which the delay actually occurs.

The phase adjustment of the LO compensates for the delay having been inserted at the IF , rather than at the RF.

A Side Benefit of Downconversion

The downconversion interferometer allows us to independently track the interferometer phase, separate from the delay compensation.

Note there are now three 'centers' in interferometry:
Sensor (antenna) pointing center
Delay (coherence) center
Phase tracking center.

All of these which are normally at the same place – but are not (aint) necessarily so.

Geometry – 2-D and 3-D Representations

To give better understanding, we now specify the geometry.

Case A: A 2-dimensional measurement plane.

Let us imagine the measurements of Vn(**b**) to be taken entirely on a plane.

Then a considerable simplification occurs if we arrange the coordinate system so one axis is normal to this plane.

Let (u,v,w) be the coordinate axes, with w normal to this plane. All distances are measured in wavelengths.

$$\mathbf{b} = (\lambda \mathrm{u}, \lambda \mathrm{v}, \lambda \mathrm{w}) = (\lambda \mathrm{u}, \lambda \mathrm{v}, 0)$$

The components of the unit direction vector, **s**, are:

$$\mathbf{s} = (l, m, n) = \left(l, m, \sqrt{1 - l^2 - m^2}\right)$$

Direction Cosines

The unit direction vector **s** is defined by its projections (l,m,n) on the (u,v,w) axes. These components are called the **Direction Cosines**.

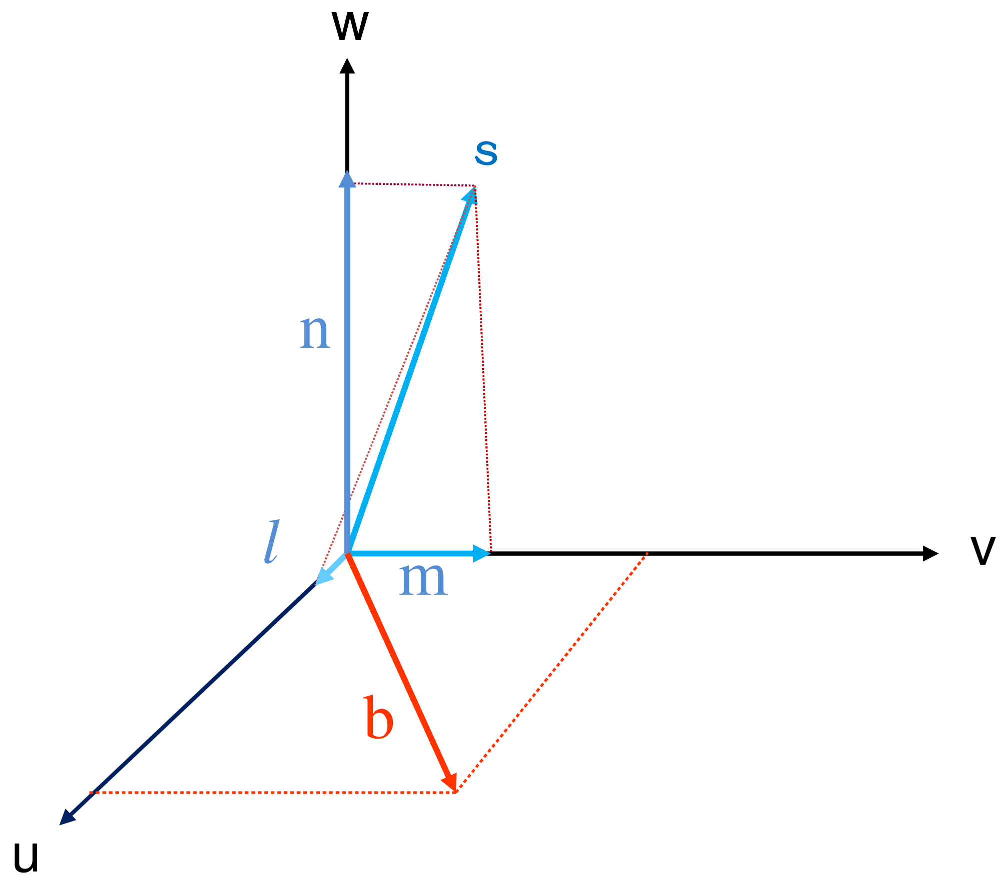
$$l = \cos(\alpha)$$

$$m = \cos(\beta)$$

$$n = \cos(\theta) = \sqrt{1 - l^2 - m^2}$$

The baseline vector **b** is specified by its coordinates (u,v,w) (measured in wavelengths). In this special case,

$$\mathbf{b} = (\lambda \mathrm{u}, \lambda \mathrm{v}, 0)$$



The 2-d Fourier Transform

Then, $\nu \mathbf{b.s}/c = ul + vm + wn = ul + vm$, from which we find,

$$V_v(u,v) = \iint \frac{I_\upsilon(l,m)}{\sqrt{1-l^2-m^2}} e^{-i2\pi(ul+vm)}\, dl dm$$

which is a **2-dimensional Fourier transform** between the projected brightness and the spatial coherence function (visibility):

$$I_v(l,m)/\cos(\theta) \Leftrightarrow V(u,v)$$

And we can now rely on a century of effort by mathematicians on how to invert this equation, and how much information we need to obtain an image of sufficient quality. Formally,

$$I_v(l,m) = \cos(\theta) \iint V_v(u,v) e^{i2\pi(ul+vm)} du dv$$

With enough measures of V, we can derive an estimate of *I*.

Theory

All this is just a restatement of the van Cittert-Zernike theorem:

The cross-correlation of the electric field on the image plane (here on the ground) is the Fourier transform of the radiation intensity distribution (the image on the sky)

-for more information read Thompson Moran & Swenson

Interferometers with 2-d Geometry

Which interferometers can use this special geometry?

a) Those whose baselines, over time, lie on a plane (any plane).

All E-W interferometers are in this group. For these, the w-coordinate points to the NCP.

WSRT (Westerbork Synthesis Radio Telescope)

ATCA (Australia Telescope Compact Array)

Cambridge 5km telescope (almost).

b) Any coplanar 2-dimensional array, at a single instance of time.

VLA or GMRT in snapshot (single short observation) mode.

What's the 'downside' of 2-d arrays?

Full resolution is obtained only for observations that are in the w-direction.

E-W interferometers have no N-S resolution for observations at the celestial equator.

A VLA snapshot of a source will have no 'vertical' resolution for objects on the horizon.

3-D Interferometers

Case B: A 3-dimensional measurement volume:

What if the interferometer does not measure the coherence function on a plane, but rather does it through a volume? In this case, we adopt a different coordinate system. First we write out the full expression:

$$V_\nu(\mathrm{u},\mathrm{v},\mathrm{w}) = \iint \frac{I_\nu(l,m)}{\sqrt{1-l^2-m^2}} e^{-2i\pi(\mathrm{u}l+\mathrm{v}m+\mathrm{w}n)} \, dl\, dm$$

(Note that this is **not** a 3-D Fourier Transform).

$$n = \cos\theta \cong 1 - \theta^2/2$$

Then, orient the coordinate system so that the w-axis points to the center of the region of interest, u points east and v north, and make use of the small angle approximation:

3-D to 2-D

With this choice, the relation between visibility and intensity becomes:

$$V_\nu(u,v) = \iint \frac{I_\nu(l,m)}{\sqrt{1-l^2-m^2}} e^{-2i\pi[ul+vm+w(\sqrt{1-l^2-m^2}-1)]} dldm$$

The third term in the phase can be neglected if it is much less than unity:

$$w\left[1-\sqrt{1-l^2-m^2}\right] << 1$$

Now, as $\cos\theta = \sqrt{1-l^2-m^2}$ is the polar angle from the delay center,

$$\theta_{\max} < \sqrt{\frac{1}{\mathrm{w}}} \le \sqrt{\frac{\lambda}{\mathrm{B}}} \sim \sqrt{\theta_{syn}}$$ (angles in radians!)

If this condition is met, then the relation between the Intensity and the Visibility again becomes a 2-dimensional Fourier transform:

$$V_\nu^{'}(u,v) = \iint \frac{I_\nu(l,m)}{\sqrt{1-l^2-m^2}} e^{-2i\pi(ul+vm)} dldm$$

The Problem with Non-coplanar Baselines

Use of the 2-D transform for non-coplanar interferometer arrays (like the VLA) always result in an error in the images.
Formally, a 3-D transform can be constructed to handle this problem – see the textbook for the details.
The errors increase inversely with array resolution, and quadratically with image field of view.
For interferometers whose field-of-view is limited by the primary beam, low-frequencies are the most affected.
The dimensionless parameter $\lambda B/D^2$ is critical:

If $\lambda B/D^2 \geq 1$ --- you've got trouble

Coverage of the U-V Plane

Obtaining a good image of a source requires adequate 'coverage' of the (u,v) plane.
To describe the (u,v) coverage, adopt an earth-based coordinate grid to describe the antenna positions:

X points to $H=0, \delta=0$ (intersection of meridian and celestial equator)

Y points to $H = -6$, $\delta = 0$ (to east, on celestial equator)

Z points to $\delta= 90$ (to NCP).

Then denote by (Bx, By, Bz) the coordinates, measured in wavelengths, of a baseline in this earth-based frame.

(Bx, By) are the projected coordinates of the baseline (in wavelengths) on the equatorial plane of the earth.
By is the East-West component
Bz is the baseline component up the Earth's rotational axis.

(U,V) Coordinates

Then, it can be shown that

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \sin H_0 & \cos H_0 & 0 \\ -\sin\delta_0 \cos H_0 & \sin\delta_0 \sin H_0 & \cos\delta_0 \\ \cos\delta_0 \cos H_0 & -\cos\delta_0 \sin H_0 & \sin\delta_0 \end{pmatrix} \begin{pmatrix} B_X \\ B_Y \\ B_Z \end{pmatrix}$$

The u and v coordinates describe E-W and N-S components of the projected interferometer baseline.

The w coordinate is the delay distance, in wavelengths between the two antennas. The geometric delay, τ_g is given by

$$\tau_g = \frac{\lambda}{c} w = \frac{w}{\upsilon}$$

Its derivative, called the fringe frequency ν_F is

$$\nu_F = \frac{dw}{dt} = -\omega_E u \cos\delta_0$$

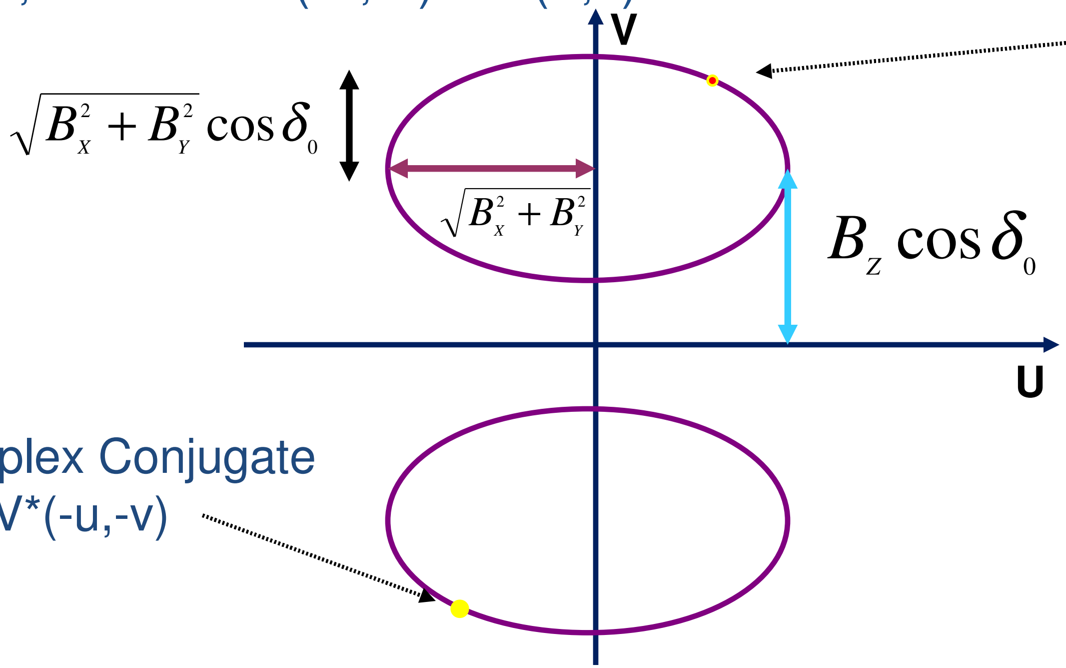
Baseline Locus

Each baseline, over 24 hours, traces out an ellipse in the (u,v) plane:

$$u^2 + \left(\frac{v - B_Z \cos\delta_0}{\sin\delta_0} \right)^2 = B_X^2 + B_Y^2$$

Because brightness is real, each observation provides us a second point, where: V*(-u,-v) = V(u,v)

A single Visibility: V(u,v)

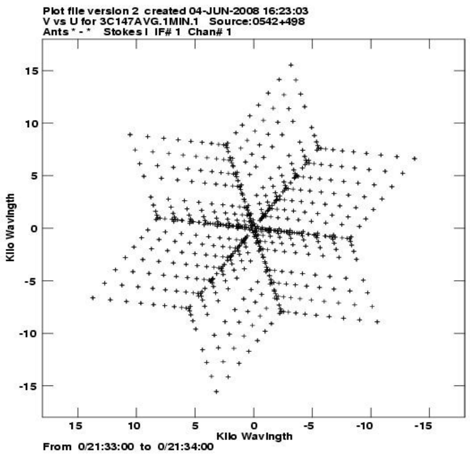


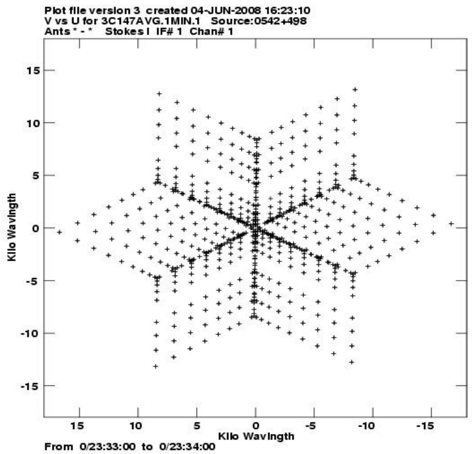
Its Complex Conjugate V*(-u,-v)

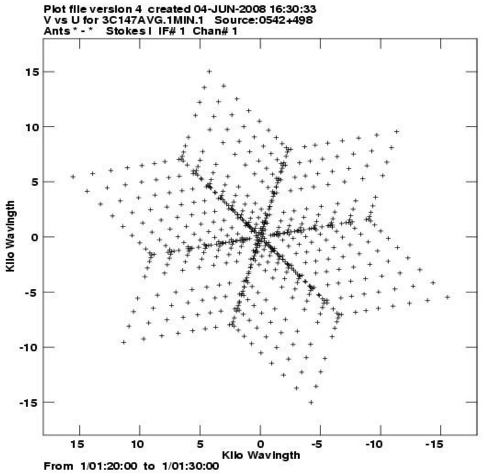
Good UV Coverage requires many simultaneous baselines amongst many antennas, or many sequential baselines from a few antennas.

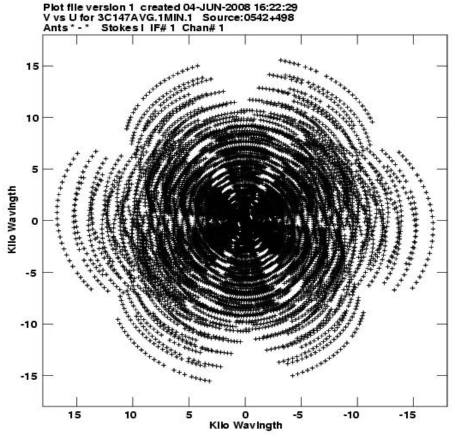
VLA (U,V) plots for 3C147 (δ = 50)

Snapshot (u,v) coverage for HA = -2, 0, +2









Coverage over all four hours.

Complications - what could possibly go wrong!

In order of appearance:

Near-field effects (in solar system)
Earth orientation: Polar motion and earth rotation
Ionosphere: - Faraday Rotation, refraction, scintillation (long λ)
Troposphere: refraction, absorption, emission (short λ)
Relativistic: 'retarded baseline'
Antenna: off-axis effects, dipoles and feed
Receiver: gain and phase errors
Electronics: bandpasses and internal delay

… etc

Summary

In this necessarily shallow overview, we have covered:
The establishment of the relationship between interferometer visibility measurement and source brightness.
The situations which permit use of a 2-D F.T.
The restrictions imposed by finite bandwidth and averaging time.
How ‘real’ interferometers track delay and phase.
The standard coordinate frame used to describe the baselines and visibilities
The coverage of the (u,v) plane.
Later lectures will discuss calibration, editing, inverting, and deconvolving these data.